Random basis algorithm for regular matroids

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We propose to investigate generating truly random bases for regular matroids (or totally unimodular matrices). The corresponding problem for the special case of graphic matroids (i.e. the problem of finding a random spanning tree of a given graph) has been extensively studied in the literature and there are several known types of algorithms: determinant based algorithms (e.g. [Gué83, CMN96, Kul90]), and random walk based algorithms (e.g. [Bro89, Ald90, Wil96, KM09] and [LP11, Chapter 4]). The notion of a "vertex" is crucial in all these approaches, and it is not clear how to generalize these techniques to non-graphic matroids.

For the more general case of regular matroids the only known technique is to start with an arbitrary basis and to perform a random walk on the "graph of bases" (where two bases are adjacent if one can be obtained from the other by an "edge-exchange" operation). Whether this chain mixes rapidly was posed as a question by Aldous in [Ald83] and it was affirmatively answered for regular matroids by Dyer and Frieze in [DF94]. The output of their algorithm is an "almost uniformly generated" basis. To our knowledge, this has been the state of art for almost two decades.

Recently, Matthew Baker and the author gave a new deterministic polynomial time algorithm for choosing random spanning trees in graphs in [BS12]. The idea behind the algorithm is very simple; associated to any graph, there is a well-defined finite abelian group called the *Jacobian*, whose order is equal to the number of spanning trees of the graph. The first step in the algorithm is to compute a presentation of the Jacobian group as a direct sum of cyclic groups. Once the group is presented in this way, it is clear how to select a random element from the group. Having done so, one can efficiently compute an explicit bijection between the group and the set of spanning trees.

We propose to carry out the same program for regular matroids. Indeed, the author is able to define the “Jacobian” group for regular matroids and to prove that the size of this group is equal to the number of bases. Further, there is a natural bijection between the group elements and the bases of the matroid. At this time our description of this bijection is geometric and not very combinatorial. The remaining challenge is to either show that the bijection is efficiently computable, or to come up with a different efficiently computable bijection.

We note that once this final step is accomplished, one would automatically obtain the desired algorithm for generating a truly random basis for regular matroids.

References

[Ald83] D. J. Aldous. Random walks on finite groups and rapidly mixing Markov chains. In *Seminaire de Probabilites XVII*, pages 243–297. 1983. Lecture Notes in Math. 986.

[Ald90] D. J. Aldous. The random walk construction of uniform spanning trees and uniform labelled trees. *SIAM J. Discrete Math.*, 3(4):450–465, 1990.

[Bro89] A. Broder. Generating random spanning trees. *Foundations of Computer Science, 30th Annual IEEE Symposium on*, pages 442–447, 1989.

[BS12] M. Baker and F. Shokrieh. Chip-firing games, potential theory on graphs, and spanning trees. Submitted, Preprint available at `arXiv:107.1313`, 34 pages, 2012.

[CMN96] C. J. Colbourn, W. J. Myrvold, and E. Neufeld. Two algorithms for unranking arborescences. *J. Algorithms*, 20(2):268–281, 1996.

[DF94] M. Dyer and A. Frieze. Random walks, totally unimodular matrices, and a randomised dual simplex algorithm. *Mathematical Programming*, 64:1–16, 1994. 10.1007/BF01582563.

[Gué83] A. Guénoche. Random spanning tree. *J. Algorithms*, 4(3):214–220, 1983.

[KM09] J. A. Kelner and A. Madry. Faster generation of random spanning trees. In *Foundations of Computer Science, 2009., 50th Annual Symposium on*, pages 13–21, Oct 2009.

[Kul90] V. G. Kulkarni. Generating random combinatorial objects. *J. Algorithms*, 11(2):185–207, 1990.

[LP11] R. Lyons and Y. Peres. *Probability on Trees and Networks*. Cambridge University Press, 2011. In preparation. Current version available at http://mypage.iu.edu/˜rdlyons/.

[Wil96] D. B. Wilson. Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996)*, pages 296–303, New York, 1996. ACM.