Application for ARC Fellowship, Spring 2013
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Title: Well-quasi-ordering graphs by the immersion relation.

One of the prominent and well-known consequences of Robertson and Seymour's Graph Minors project is that every minor-closed property can be characterized by finitely many graphs. In the (currently) last paper of the Graph Minors series, Robertson and Seymour prove the same for properties closed under weak immersion [2]. As a result, if the propery $\mathcal{Q}$ of graphs is closed under deleting edges and splitting off two edges which have exactly one common end, then there exist finitely many graphs $H_1, H_2, ..., H_n$ (only depending on $\mathcal{Q}$) such that the decision problem of whether the input graph satisfies $\mathcal{Q}$ can be solved in polynomial time by checking whether the input graph admits a weak H_i-immersion for every $1 \leq i \leq n$. Fellows and Langston [1] used this fact to deduce polynomial time algorithms on many problems that were not known to be in **P**, and gave more efficient algorithms for some other problems. For example, in the area of VLSI design, the fixed-parameter instances of the min cut linear arrangement problem and 2-D grid load factor problem can be decided in quadratic time.

Nash-Williams conjectured in the 1960's that the relations of weak immersion and strong immersion are well-quasi-orderings of finite graphs. In other words, given an infinite sequence $G_1, G_2, ...$ of finite graphs, there exist $i < j$ such that G_j admits a weak (strong, respectively) G_i-immersion. The conjecture on weak immersion was proved via very complicated arguments [2], but the conjecture on strong immersion, which implies the weak immersion conjecture, is still open. Furthermore, the conjecture on strong immersion implies that every strong immersion-closed property can be decided in polynomial time. Note that the class of strong immersion-closed properties strictly contains the class of weak immersion-closed properties. The objective of this project is to develop structural theorems about graph immersions, and use these to prove the strong immersion conjecture or at least to give a more laconic proof for the weak immersion conjecture. In comparison to the fact that structural theorems about graph minors were widely explored, only few about graph immersions could be found in the literature. As structural theorems about graph minors lead to great successes in deriving many algorithmic results, it is reasonable to expect the same for graph immersions. On the other hand, the known proof of the weak immersion conjecture does not use structural theorems, so our approach will lead an essentially different proof.

For formal definitions, we say that a graph G admits a *weak H-immersion* if there are an injective map $\pi_V : V(H) \to V(G)$ and a map π_E that maps each edge xy of H to a path in G with ends $\pi_V(x)$ and $\pi_V(y)$ such that $\pi_E(e_1)$ and $\pi_E(e_2)$ are edge-disjoint for every two different edges e_1 and e_2 of H. The vertices in the image of π_V are called *branch vertices*. We say that G admits a *strong H-immersion* if G admits a weak H-immersion such that every branch vertex is not an internal vertex of $\pi_E(e)$ for any edge e in H. The concept of immersions can be alternatively defined. Given two edges e_1 and e_2 in G, where x_i and y_i are the ends of e_i for $i = 1, 2$ such that $x_1 = x_2$ but $y_1 \neq y_2$, *splitting off* e_1 and e_2 is the operation that deletes e_1 and e_2 from G and adds an edge with ends y_1, y_2. Then G admits a weak H-immersion if H can be obtained from a subgraph of G by consecutively splitting off two edges that have exactly one common end, and deleting edges and isolated vertices. Similarly, G admits a strong H-immersion if H can be obtained from a subgraph of G by consecutively picking a vertex, pairing all edges incident with this vertex, and splitting off each pair of these edges, and deleting edges and isolated vertices.

We currently are able to prove the following theorem that gives structure of graphs that do not immerse a family of graphs. (The theorem needs a couple of definitions to be precisely stated, so we only give an informal description here.)

Theorem. *Let h and k be positive intgers. Then there exist $f(h,k)$ such that every graph G satisfies one of the following.*

1. *G weakly immerses any graph of order h and maximum degree at most k.*

2. *The tree-cut width of G is at most $f(h,k)$.*

3. *For every edge-tangle of G of order at least $f(h,k)$, there exist $f(h,k)$ edges in G such that after deleting these edges from G, either every vertex of degree at least k is separated from the edge-tangle by at most $k-1$ edges, or G can be "nearly" drawn on a surface of genus $f(h,k)$ with respect to this edge-tangle such that every vertex "strictly" lying on the surface has degree at most three.*

The concept of tree-cut width was proposed by Seymour and Wollan [3], and it is an analogue of tree-width. And one can think that an edge-tangle is more or less a subgraph of high edge-connectivity.

Our approach toward the conjectures on immersions is proof by contradiction. Suppose that there is an infinite sequence of graphs such that no graph in the sequence admits an immersion of another graph in the sequence of smaller index. So every graph in this sequence cannot immerse the first graph in the sequence, and hence our theorem applies.

The main objectives of this project are first to boost the above theorem to strong immersions, and then showing that the relation of strong immersion is a well-quasi-ordering on graphs of bounded tree-cut width. Once the above objectives are achieved, we can assume that all graphs in the bad sequence satisfy statement 3 of the above theorem. In addition, whenever the first case in statement 3 happens, G can be written as an edge-sum of graphs of either smaller order or maximum degree at most $k-1$. In other words, by repeatedly applying our theorem, one can reduce the order or the maximum degree of the pieces of graphs in the bad sequence until the second case of statement 3 happens. Hence, it is sufficient to prove that the relation of strong immersion is a well-quasi-ordering on graphs which have a tree-structure such that each piece can be nearly drawn on a surface of bounded genus after deleting a bounded number of edges.

In summary, structural theorems on graph immerions could lead many algorithmic results as in the case of graph minors, but only limited results are known. We propose to build structural theorems on graph immersions, and use them to prove Nash-Williams' conjecture on graph immersions. Once these conjectures are confirmed, the existence of polynomial-time algorithms for some potential problems will be derived.

References

[1] M.R. Fellows and M.A. Langston, *On well-partial-order theory and its application to combinatorial problems of VLSI design*, SIAM J. Discrete Math. 5 (1992), 117–126.

[2] N. Robertson, P.D. Seymour, *Graph minors XXIII: The Nash-Williams immersion conjecture,* J. Combin. Theory, Ser. B **100** (2010) 181–205.

[3] P.D. Seymour and P. Wollan, *The structure of graphs no admitting a fixed immersion,* manuscript.